Estimation of the Reliability Function for Four-Parameter Exponentiated Generalized Lomax distribution

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Abstract: Problem Statement: The four-parameter exponentiated generalized Lomax distribution has been introduced. The uniformly minimum variance unbiased and maximum likelihood estimation methods are the way to estimate the parameters of the distribution. In this study we explore and compare the performance of the uniformly minimum variance unbiased and maximum likelihood estimators of the reliability function R(t)=P(X>t) and P=P(X>Y) for the four-parameter exponentiated generalized Lomax distribution. **Approach:** A new technique of obtaining these parametric functions is introduced in which major role is played by the powers of the parameter(s) and the functional forms of the parametric functions to be estimated are not needed. We explore the performance of these estimators numerically under varying conditions.

Keywords: four-parameter exponentiated generalized Lomax distribution, uniformly minimum variance unbiased estimators (UMVUES), maximum likelihood estimators (MLES), bootstrap method.

1. INTRODUCTION

The reliability function R(t) is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item, then R(t). Another measure of reliability under stress-strength set-up is the probability P(X>Y), which represents the reliability of an item of random strength X subject to random stress Y. Many researchers have considered the problems of estimation of R(t) and 'P' for various lifetime distributions, like, exponential, gamma, Weibull, half-normal, Maxwell, Rayleigh, Burr and others. For a brief review, one may refer to [14], [11], [16], [3], [18], [6], [19], [5] and others.

Statistical distributions are very useful in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. As a result, many new distributions have been developed and studied. For example, [8] proposed a generalization of the standard exponential distribution, called the exponentiated exponential distribution, defined by the cumulative distribution function (cdf)

$$F(x;\alpha,\lambda)=\left(1-e^{-\lambda x}\right)^{\alpha-1};\ x,\alpha,\lambda>0.$$

This equation is simply the αth power of the standard exponential cdf. For a full discussion and some of its mathematical properties, see [9].

[8] introduced exponentiated generalized class of distributions by taking a continuous cdf G(x), say, as

$$F(x;\alpha,\beta)=\left[1-\left\{1-G(x)\right\}^{\beta}\right]^{\alpha}, \tag{1.1}$$

where α>0 and β>0 are two additional shape parameters and G(x) is known as the baseline distribution. The probability density function (pdf) of the new class has the form

$$f(x;\alpha,\beta)=\alpha\beta\left\{1-G(x)\right\}^{\beta-1}\left[1-\left\{1-G(x)\right\}^{\beta}\right]^{\alpha}g(x), \tag{1.2}$$

where g(x) is the pdf corresponding to the cdf G(x).

The exponentiated generalized family of distributions (1.2) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology and it extends several well-known distributions in the literature. Note that even if g(x) is a symmetric distribution, the distribution f(x) will not be a symmetric distribution. The two extra parameters in (1.2) can control both tail weights and possibly adding entropy to the center of the exponentiated generalized family of distributions. In particular, if g(x) and G(x) represent the pdf and cdf, respectively, of two parameter Lomax distribution, then

$$g(x;\lambda,\delta)=\lambda\delta\left[1+\delta x\right]^{-(\lambda+1)} \tag{1.3}$$

and

$$G(x;\lambda,\delta)=1-\left[1+\delta x\right]^{-\lambda}. \tag{1.4}$$

From (1.1), (1.2), (1.3) and (1.4), we get

$$f(x;\alpha,\beta,\lambda,\delta)=\alpha\beta\lambda\delta(1+\delta x)^{-(\lambda+1)}\left\{(1+\delta x)^{-\lambda}\right\}^{\beta-1}\cdot\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\alpha-1};\ x,\alpha,\beta,\lambda,\delta>0 \tag{1.5}$$

and

$$F(x;\alpha,\beta,\lambda,\delta)=\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\alpha};\ x,\alpha,\beta,\lambda,\delta>0, \tag{1.6}$$

where $f(x;\alpha,\beta,\lambda,\delta)$ and $F(x;\alpha,\beta,\lambda,\delta)$, respectively, represents the pdf and cdf of exponentiated generalized Lomax distribution.

The graphs of exponentiated generalized Lomax distribution for various values of α, β, λ and δ are given in Fig.1. The figure shows that the density function can take different shapes for different values of these parameters.

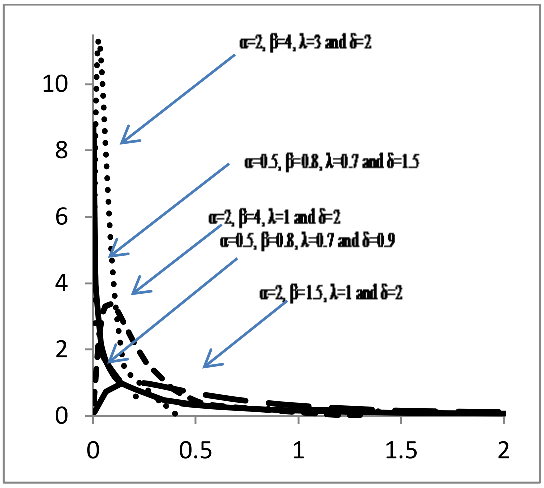

Fig. 1: Curves of $f(x;\alpha,\beta,\lambda,\delta)$.

Hazard function: The hazard function of exponentiated exponential-Weibull distribution using Equations (1.5) and (1.6) is given by

$$h(x;\alpha,\beta,\lambda,\delta)=\frac{\alpha\beta\lambda\delta(1+\delta x)^{-(\lambda+1)}\left\{(1+\delta x)^{-\lambda}\right\}^{\beta-1}}{1-\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\alpha}} \quad (1.7)$$

$$\cdot\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\alpha-1};\ x,\ \alpha,\ \beta,\ \lambda,\ \delta>0.$$

Hazard functions for various values of α, β, λ and δ are given in Fig.2.

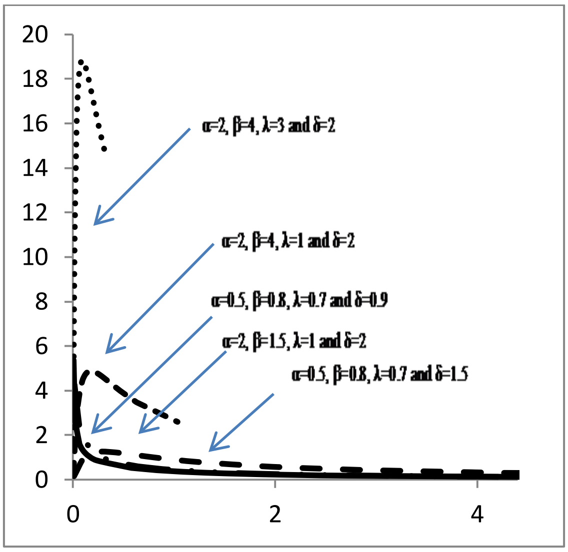

Fig.2: Curves of Hazard function.

The purpose of the present paper is manifold. We consider exponentiated generalized Lomax distribution. The UMVUES and MLES of the reliability function R(t) and 'P' are derived for the complete sample case. In order to obtain these estimators, the major role is played by the UMVUES and MLES of the powers of parameter(s) and the functional forms of the parametric functions to be estimated are not needed. Simulation study is carried out to investigate the performances of these estimators. A comparative study of different methods of estimation is done.

In Section 2, we derive the UMVUES of the reliability function R(t) and 'P' assuming α to be unknown but other parameters are known. In Section 3, we obtain MLES of the reliability function R(t) and 'P', when all the parameters are unknown. In Section 4, simulation study is performed. In Section 5, discussion is made. Finally, in Section 6, conclusions are given.

2. UMVUES OF THE POWERS OF α, R(t) AND 'P' WHEN β, λ AND δ ARE KNOWN

Let $X_1, X_2, ..., X_n$ be a random sample of size n from (1.5).

Lemma 1: Let $S=-\sum_{i=1}^{n}\ln\left[1-\left\{(1+\delta X_i)^{-\lambda}\right\}^{\beta}\right]$ Then, S is complete and sufficient for the distribution given at (1.5). Moreover, the pdf of S is

$$h(s;\alpha)=\frac{\alpha^n}{\Gamma(n)}s^{n-1}e^{-\alpha s};\ 0<s<\infty.$$

Proof: From (1.5), the joint pdf of $X_1, X_2, ..., X_n$ is

$$h(x_1,x_2,...,x_n;\alpha,\beta,\lambda,\delta)=(\alpha\beta\lambda\delta)^n$$

$$\cdot\prod_{i=1}^{n}\left[\frac{(1+\delta X_i)^{-(\lambda+1)}\left\{(1+\delta X_i)^{-\lambda}\right\}^{\beta-1}}{1-\left\{(1+\delta X_i)^{-\lambda}\right\}^{\beta}}\right]\exp(-\alpha S) \quad (2.1)$$

It follows from (2.1) and Fishers-Neyman factorization theorem [see Rohatgi (1976, p. 341)] that T is sufficient for the $f(x;\alpha,\beta,\lambda,\delta)$ It is easy to see that the rv $V=-2\alpha\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)$ follows $\chi^2_{(2)}$ distribution. Thus, from the well-known additive property of Chi-square distribution [see [12]], $-2\alpha S=-2\alpha\sum_{i=1}^{n}\ln\left(1-\left\{(1+\delta X_i)^{-\lambda}\right\}^{\beta}\right)$ is a $\chi^2_{(2n)}$ rv and the result follows. Since the distribution of T belongs to exponential family, it is also complete [see [15]].

The following theorem provides the UMVUES of the powers of α.

Theorem 1: For $q \in (-\infty, \infty)$, the UMVUE of α^q is

$$\hat{\alpha}^q = \begin{cases} \dfrac{\Gamma(n)}{\Gamma(n-q)} S^{-q}; & q<n, \\ 0 & ; \text{otherwise.} \end{cases}$$

Proof: From (2.2),

$$E(2\alpha S)^{-q} = E[\chi^2_{(2n)}]^{-q}$$

$$= \frac{\Gamma(n-q)}{2^q \Gamma(n)}; \ q<n,$$

or,

$$E\left[S^{-q} \left\{\frac{\Gamma(n)}{\Gamma(n-q)}\right\}\right] = \alpha^q$$

Hence, the theorem follows from Lehmann-Scheffé Theorem [see [15]].

In the following lemma, we provide the UMVUE of the sampled pdf (1.5) at specified point 'x'.

Lemma 2: The UMVUE of $f(x;\alpha,\beta,\lambda,\delta)$ at a specified point 'x' is

$$\hat{f}(x;\alpha,\beta,\lambda,\delta) = (n-1)\beta\lambda\delta\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)^{-1}(1+\delta X)^{-(\lambda+1)}$$

$$\left\{(1+\delta X)^{-\lambda}\right\}^{\beta-1}\left[1+S^{-1}\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)\right]^{n-2}$$

$$; -S<\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)$$

Proof: We can write (1.5) as

$$f(x;\alpha,\beta,\lambda,\delta) = \beta\lambda\delta\left(1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right)^{-1}$$

$$\cdot(1+\delta x)^{-(\lambda+1)}\left\{(1+\delta x)^{-\lambda}\right\}^{\beta-1} \qquad (2.2)$$

$$\sum_{i=1}^{\infty}\frac{\left[\ln\left(1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right)\right]^{i}}{i!}\alpha^{i+1}$$

Using (2.2), Theorem 1 and Lemma 1 of Chaturvedi and Tomer (2002), UMVUE of $f(x;\alpha,\beta,\lambda,\delta)$ at a specified point 'x' is

$$\hat{f}(x;\alpha,\beta,\lambda,\delta) = \beta\lambda\delta\left(1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right)^{-1}(1+\delta x)^{-(\lambda+1)}$$

$$\cdot\left\{(1+\delta x)^{-\lambda}\right\}^{\beta-1}\sum_{i=1}^{\infty}\frac{\left[\ln\left(1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right)\right]^{i}}{i!}\hat{\alpha}^{i+1}$$

$$= (n-1)\beta\lambda\delta S^{-1}\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)^{-1}(1+\delta X)^{-(\lambda+1)}$$

$$\cdot\left\{(1+\delta X)^{-\lambda}\right\}^{\beta-1}\left[1+S^{-1}\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)\right]^{n-2}$$

$$; -S<\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)$$

and the lemma holds.

In the following theorem, we obtain UMVUE of R(t).

Theorem 2: The UMVUE of R(t) is given by

$$\hat{R}(t) = 1-\left[1+S^{-1}\ln\left(1-\left\{(1+\delta t)^{-\lambda}\right\}^{\beta}\right)\right]^{n-1}$$

$$; -S<\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)$$

Proof: Let us consider the expected value of the integral $\int_t^{\infty}\hat{f}(x;\alpha,\beta,\lambda,\delta)dx$, with respect to S, i.e.,

$$\int_0^{\infty}\left\{\int_t^{\infty}\hat{f}(x;\alpha,\beta,\lambda,\delta)dx\right\}h(s;\alpha)ds$$

$$= \int_t^{\infty}[E_S\{\hat{f}(x;\alpha,\beta,\lambda,\delta)\}]dx$$

$$= \int_t^{\infty} f(x;\alpha,\beta,\lambda,\delta)dx$$

$$= R(t). \qquad (2.3)$$

We conclude from (2.3) that the UMVUE of R(t) can be obtained simply by integrating $\hat{f}(x;\alpha,\beta,\lambda,\delta)$ from t to ∞. Thus, from Lemma 2,

$$\hat{R}(t) = (n-1)S^{-1}\beta\lambda\delta\int_t^{\infty}\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)^{-1}(1+\delta X)^{-(\lambda+1)}$$

$$\cdot\left\{(1+\delta X)^{-\lambda}\right\}^{\beta-1}dx\left[1+S^{-1}\ln\left(1-\left\{(1+\delta X)^{-\lambda}\right\}^{\beta}\right)\right]^{n-2}$$

$$= (n-1)\int_{S^{-1}\ln\left(1-\left\{(1+\delta t)^{-\lambda}\right\}^{\beta}\right)}^{0}(1+z)^{n-2}dz$$

and the theorem follows.

Let X and Y be two independent rvs following the families of distributions $f_1(x;\alpha_1,\beta_1,\lambda_1,\delta_1)$ and $f_2(y;\alpha_2,\beta_2,\lambda_2,\delta_2)$, respectively. We assume that α_1 and α_2 are unknown but $\beta_1, \beta_2, \lambda_1, \lambda_2, \delta_1$ and δ_2 are known. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $f_1(x;\alpha_1,\beta_1,\lambda_1,\delta_1)$ and $Y_1, Y_2, \ldots, Y_m$ be a random sample of size m from $f_2(y;\alpha_2,\beta_2,\lambda_2,\delta_2)$.

Let us denote by $S=-\sum_{i=1}^{n}\ln\left[1-\left\{(1+\delta_1 x_i)^{-\lambda_1}\right\}^{\beta_1}\right]$ and $S=-\sum_{i=1}^{n}\ln\left[1-\left\{(1+\delta_1 x_i)^{-\lambda_1}\right\}^{\beta_1}\right]$. In what follows, we

In what follows, we obtain the UMVUE of 'P'.

Theorem 3: The UMVUE of 'P' is given by

$$\hat{P}=\begin{cases} 1-(m-1)\displaystyle\int_0^{-T^{-1}\ln\left[1-\left\{\left(1-\frac{\delta_2}{\delta_1}\left[1-\left\{\left(1-e^{-S}\right)^{1/\beta_1}\right\}^{-1/\lambda_1}\right]\right)^{-\lambda_2}\right\}^{\beta_2}\right]} \left[1+S^{-1}\ln\left[1-\left\{\left(1-\frac{\delta_1}{\delta_2}\left[1-\left\{\left(1-e^{-Tv}\right)^{1/\beta_2}\right\}^{-1/\lambda_2}\right]\right)^{-\lambda_1}\right\}^{\beta_1}\right]\right]^{n-1} \\ \qquad \cdot(1-v)^{m-2}dv;\ \delta_1^{-1}\left[\left(1-\left\{e^{-S}\right)^{1/\beta_1}\right\}^{-\lambda_1^{-1}}-1\right]<\delta_2^{-1}\left[\left(1-\left\{e^{-T}\right)^{1/\beta_2}\right\}^{-\lambda_2^{-1}}-1\right], \\ 1-(m-1)\displaystyle\int_0^1\left[1+S^{-1}\ln\left[1-\left\{\left(1-\frac{\delta_1}{\delta_2}\left[1-\left\{\left(1-e^{-Tv}\right)^{1/\beta_2}\right\}^{-1/\lambda_2}\right]\right)^{-\lambda_1}\right\}^{\beta_1}\right]\right]^{n-1} \\ \qquad \cdot(1-v)^{m-2}dv;\ \delta_1^{-1}\left[\left(1-\left\{e^{-S}\right)^{1/\beta_1}\right\}^{-\lambda_1^{-1}}-1\right]>\delta_2^{-1}\left[\left(1-\left\{e^{-T}\right)^{1/\beta_2}\right\}^{-\lambda_2^{-1}}-1\right]. \end{cases}$$

Proof: From the arguments similar to those adopted in proving Theorem 1, it can be shown that the UMVUE of 'P' is given by

$$\hat{P}=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\hat{f}(x;\alpha_1,\beta_1,\lambda_1,\delta_1)\hat{f}(y;\alpha_2,\beta_2,\lambda_2,\delta_2)dx\,dy$$

$$=\int_{y=0}^{\infty}\hat{R}(y;\alpha_1,\beta_1,\lambda_1,\delta_1)\hat{f}(y;\alpha_2,\beta_2,\lambda_2,\delta_2)dy$$

$$=1-(m-1)\beta_2\lambda_2\delta_2T^{-1}\int_{y=0}^{\max\left\{\delta_1^{-1}\left[\left(1-\left\{e^{-S}\right)^{1/\beta_1}\right\}^{-\lambda_1^{-1}}-1\right],\ \delta_2^{-1}\left[\left(1-\left\{e^{-T}\right)^{1/\beta_2}\right\}^{-\lambda_2^{-1}}-1\right]\right\}}\left[1+S^{-1}\ln\left(1-\left\{(1+\delta_1Y)^{-\lambda_1}\right\}^{\beta_1}\right)\right]^{n-1}$$

$$\cdot\left(1-\left\{(1+\delta_2Y)^{-\lambda_2}\right\}^{\beta_2}\right)^{-1}\left[1+T^{-1}\ln\left(1-\left\{(1+\delta_2Y)^{-\lambda_2}\right\}^{\beta_2}\right)\right]^{m-2}dy \tag{2.4}$$

and the theorem follows on combining (2.4).

Corollary 1: In the case when $\beta_1=\beta_2=\beta$, say, $\lambda_1=\lambda_2=\lambda$, say, and $\delta_1=\delta_2=\delta$, say

$$\hat{P}=\begin{cases}1-\displaystyle\sum_{j=0}^{m-2}(-1)^j\frac{(n-1)!(m-1)!}{(n+j)!(m-j-2)!}\left(\frac{S}{T}\right)^{j+1}; & S>T,\\ 1-\displaystyle\sum_{i=0}^{n-1}(-1)^i\frac{(n-1)!(m-1)!}{(n-i-1)!(m+i-1)!}\left(\frac{T}{S}\right)^{i}; & S<T.\end{cases}$$

REMARK 1: It follows from Theorem 1 that $V(\hat{\alpha})=\frac{\alpha^2}{n-1}\to 0$ as $n\to\infty$. Thus $\hat{\theta}$ is a consistent estimator of α. Since, $\hat{f}(x;\alpha,\beta,\lambda,\delta)$, $\hat{R}(t)$ and $\hat{P}$ are continuous functions of consistent estimators, they are also consistent estimators.

3. MLES OF R(t) AND 'P' WHEN ALL THE PARAMETERS ARE UNKNOWN

Let $X_1, X_2, ..., X_n$ be a random sample of size n from (1.5). We denote by $\Theta=(\alpha, \beta, \lambda, \delta)$, where α, β, λ and δ are unknown. From (1.5), the log-likelihood function is

$$\ln L\left(\Theta|\underset{\sim}{x}\right)=n\ln\alpha+n\ln\beta+n\ln\lambda+n\ln\delta$$
$$-(\lambda+1)\sum_{i=1}^{n}\ln(1+\delta x_i)-(\beta-1)\lambda\sum_{i=1}^{n}(1+\delta x_i)$$
$$+(\alpha-1)\sum_{i=1}^{n}\ln\left[1-\left\{(1+\delta x_i)^{-\lambda}\right\}^{\beta}\right]. \tag{3.1}$$

Considering negative log-likelihood, then differentiating it with respect to all unknown parameters and equating these differential coefficients to zero and solving them simultaneously, let $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\lambda}$ and $\tilde{\delta}$ be the MLES of α, β, λ and δ, respectively.

The following lemma provides the MLE of $f(x;\alpha,\beta,\lambda,\delta)$ at a specified point 'x'.

Lemma 3: The MLE of $f(x;\alpha,\beta,\lambda,\delta)$ at a specified point 'x' is

$$\tilde{f}(x;\alpha,\beta,\lambda,\delta)=\tilde{\alpha}\tilde{\beta}\tilde{\lambda}\tilde{\delta}(1+\tilde{\delta}x)^{-(\tilde{\lambda}+1)}\left\{(1+\tilde{\delta}x)^{-\tilde{\lambda}}\right\}^{\tilde{\beta}-1}$$
$$\cdot\left[1-\left\{(1+\tilde{\delta}x)^{-\tilde{\lambda}}\right\}^{\tilde{\beta}}\right]^{\tilde{\alpha}-1}.$$

Proof: The proof follows from (1.5) and one-to-one property of the MLES.

In the following theorem, we derive the MLE of R(t).

Theorem 4: The MLE of $R(t)$ is given by

$$\tilde{R}(t)=1-\left[1-\left\{(1+\tilde{\delta}x)^{-\tilde{\lambda}}\right\}^{\tilde{\beta}}\right]^{\tilde{\alpha}}.$$

Proof: Using Lemma 3 and one-to-one property of the MLES, we get

$$\tilde{R}(t)=\int_t^{\infty}\tilde{f}(x;\alpha,\beta,\lambda,\delta)dx$$

$$=\tilde{\alpha}\int_{\left[1-\left\{(1+\tilde{\delta}t)^{-\tilde{\lambda}}\right\}^{\tilde{\beta}}\right]}^{1}u^{\tilde{\alpha}-1}du$$

and the theorem follows.

Corollary 2: For the case when β, λ and δ are known MLE of R(t) is given by

$$\tilde{R}(t)=1-\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\tilde{\alpha}}.$$

Proof: For known β, λ and δ, MLE of $f(x;\alpha,\beta,\lambda,\delta)$ at a specified point 'x' is

$$\tilde{f}(x;\alpha,\beta,\lambda,\delta)=\tilde{\alpha}\beta\lambda\delta(1+\delta x)^{-(\lambda+1)}\left\{(1+\delta x)^{-\lambda}\right\}^{\beta-1}$$
$$\cdot\left[1-\left\{(1+\delta x)^{-\lambda}\right\}^{\beta}\right]^{\tilde{\alpha}-1} \tag{3.6}$$

From the arguments similar to those adopted in proving Theorem 4 and (3.6), corollary directly follows.

In the following theorem, we obtain the MLE of 'P'.

Theorem 5: The MLE of 'P' is given by

$$\tilde{P}=\tilde{\alpha}_1\int_0^1\left[1-\left\{\left(1+\tilde{\delta}_1^{-1}\tilde{\delta}_2\left[-1+\left\{(1-v)^{1/\tilde{\beta}_1}\right\}^{-1/\tilde{\lambda}_1}\right]\right)^{\tilde{\lambda}_2}\right\}^{\tilde{\beta}_2}\right]^{\tilde{\alpha}_2}\cdot\left(1-v^{\tilde{\beta}_1}\right)^{\tilde{\alpha}_1-1}dv,$$

where $\left(\tilde{\alpha}_1,\tilde{\beta}_1,\tilde{\lambda}_1,\tilde{\delta}_1\right)$ and $\left(\tilde{\alpha}_2,\tilde{\beta}_2,\tilde{\lambda}_2,\tilde{\delta}_2\right)$ are the MLES based on $\underline{x}=\left(x_1, x_2, ..., x_n\right)$ and $\underline{y}=\left(y_1, y_2, ..., y_m\right)$, on X and Y, respectively.

Proof: Using Lemma 3, Theorem 5 and the one-to-one property of the MLES, we have

$$\tilde{P}=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\tilde{f}_1(x;\tilde{\alpha}_1,\tilde{\beta}_1,\tilde{\lambda}_1,\tilde{\delta}_1)\tilde{f}_2(y;\tilde{\alpha}_2,\tilde{\beta}_2,\lambda_2,\delta_2)dx\,dy$$

$$=\int_0^{\infty}\tilde{R}_2(x;\tilde{\alpha}_2,\beta_2,\lambda_2,\delta_2)\tilde{f}_1(x;\tilde{\alpha}_1,\beta_1,\lambda_1,\delta_1)dx$$

and the theorem holds.

Corollary 3: For the case when β, λ and δ are known MLE of 'P' is given by

$$\tilde{P}=\tilde{\alpha}_1\int_0^1\left[1-\left\{\left(1+\delta_1^{-1}\delta_2\left[-1+\left\{(1-v)^{1/\beta_1}\right\}^{-1/\lambda_1}\right]\right)^{\lambda_2}\right\}^{\beta_2}\right]^{\tilde{\alpha}_2}\cdot\left(1-v^{\beta_1}\right)^{\tilde{\alpha}_1-1}dv,$$

where $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are the MLES based on $\underline{x}=\left(x_1, x_2, ..., x_n\right)$ and $\underline{y}=\left(y_1, y_2, ..., y_m\right)$, on X and Y, respectively.

REMARKS 2:

(i) In the literature, the researchers have dealt with the estimation of $R(t)$ and 'P', separately. If we look at the proof of Theorems 2, 3, 4 and 5, we observe that the UMVUE and MLE of the sampled pdf is used to obtain the UMVUES and MLES of $R(t)$ and 'P', respectively. Thus we have established interrelationship between the two estimation problems. Moreover, in the present approach, one does not require the expressions of R(t) and 'P'.

(ii) Since the UMVUES and MLES of powers of α are obtained under same conditions, we compare their performances. For q = -1 the UMVUE and MLE of α are, respectively $\hat{\alpha}=(n-1)(-T)^{-1}$ and $\tilde{\alpha}=(n)(-T)^{-1}$. For these estimators,

$$V(\hat{\alpha})=\frac{\alpha^2}{n-2} \text{ and } V(\tilde{\alpha})=\frac{n^2\alpha^2}{(n-1)^2(n-2)}.$$ Thus,

$$V(\tilde{\alpha})-V(\hat{\alpha})=\frac{(2n-1)}{(n-1)(n-2)}\alpha^2>0.$$

Thus, the UMVUE of α is more efficient than its MLE. Similarly, we can compare the performances of these estimators for other powers of α.

4. NUMERICAL FINDINGS

In order to verify the result obtained in remark 2 (ii), we have calculated variances of $\hat{\alpha}$ and $\tilde{\alpha}$, when the other parameters are known, for samples of sizes n = 5, 10, 20, 30 and 50 corresponding to α = 0.20(0.4)3.40. These results are reported in Table 1.

Table 7.1.

n	5		10		20		30		50	
α	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$	$v(\hat{\alpha})$	$v(\tilde{\alpha})$
0.20	0.0133	0.0208	0.0050	0.0062	0.0022	0.0025	0.0014	1.1986	8e-04	9e-04
0.60	0.1200	0.1875	0.0450	0.0556	0.0200	0.0222	0.0129	0.0138	0.0075	0.0078
1.00	0.3333	0.5208	0.1250	0.1543	0.0556	0.0616	0.0357	0.0382	0.0208	0.0217
1.40	0.6533	1.0208	0.2450	0.3025	0.1089	0.1207	0.0700	0.0749	0.0408	0.0425
1.80	1.0800	1.6875	0.4050	0.5000	0.1800	0.1994	0.1157	0.1238	0.0675	0.0703
2.20	1.6133	2.5208	0.6050	0.7469	0.2689	0.2979	0.1729	0.1850	0.1008	0.1050
2.60	2.2533	3.5208	0.8450	1.0432	0.3756	0.4161	0.2414	0.2584	0.1408	0.1466
3.00	3.0000	4.6875	1.1250	1.3889	0.5000	0.5540	0.3214	0.3440	0.1875	0.1952
3.40	3.8533	6.0208	1.4450	1.7840	0.6422	0.7116	0.4129	0.4418	0.2408	0.2508

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In order to verify the consistency of the estimators obtained, we have drawn sample of sizes n = 30 from (1.5), with $\alpha = 2, \beta=4, \lambda=1$ and $\delta = 2$. In Fig. 3, we have plotted $f(x;\alpha,\beta,\lambda,\delta)$, $\hat{f}(x;\alpha,\beta,\lambda,\delta)$ and $\tilde{f}(x;\alpha,\beta,\lambda,\delta)$, respectively, corresponding to this sample. We conclude from Fig. 3 that curves of $\hat{f}(x;\alpha,\beta,\lambda,\delta)$ and $\tilde{f}(x;\alpha,\beta,\lambda,\delta)$ overlap to the curve of $f(x;\alpha,\beta,\lambda,\delta)$ for n=30. This justifies the consistency property of the estimators.

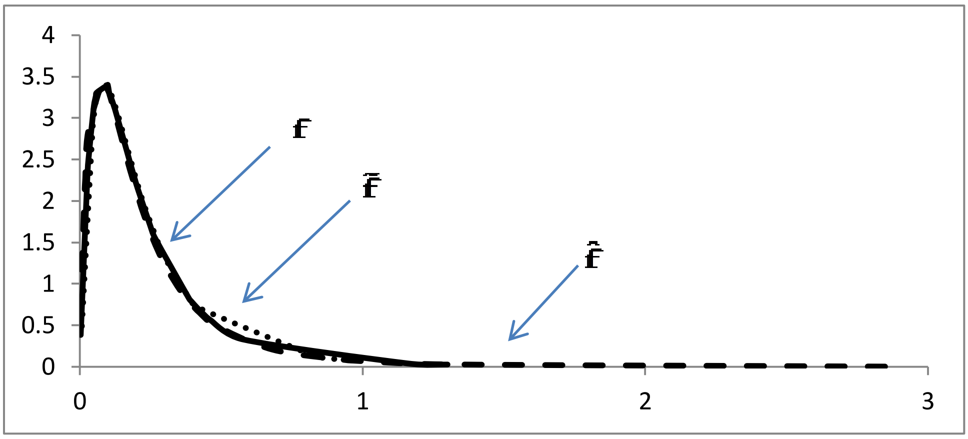

Fig. 3: Curves of $f(x;\alpha,\beta,\lambda,\delta)$, $\hat{f}(x;\alpha,\beta,\lambda,\delta)$ and $\tilde{f}(x;\alpha,\beta,\lambda,\delta)$.

In order to demonstrate the application of the theory developed in Section 3, we generated the following sample of size n=30 from (1.5) for $\alpha=2.0, \beta=4.0, \lambda=1.0$ and $\delta=2.0$.

0.0135, 0.0475, 0.0501, 0.0547, 0.0562, 0.0584, 0.0709, 0.0761, 0.0866, 0.0982, 0.0992, 0.1035, 0.1071, 0.1414, 0.1532, 0.1594, 0.1907, 0.2085, 0.2514, 0.2827, 0.2894, 0.2904, 0.3651, 0.3974, 0.3993, 0.5105, 0.7154, 0.7456, 0.7801, 0.7901.

From (3.1), we get $\tilde{\alpha}=1.838271$, $\tilde{\beta}=3.981124$, $\tilde{\lambda}=0.945446$ and $\tilde{\delta}=2.089517$. It can be seen that -2 ln L= -23.95892, R(0.05) = 0.8995 and $\tilde{R}(0.45)$ = 0.8824.

In order to obtain the MLE of 'P', we have generated one more sample of size m=30 from (1.5) for $\alpha=3.5, \beta=4.0, \lambda=1.0$ and $\delta=2.0$.

0.0416, 0.0561, 0.0948, 0.1116, 0.1224, 0.1261, 0.1395, 0.1449, 0.1618, 0.1624, 0.1762, 0.1762, 0.2039, 0.2061, 0.2267, 0.2371, 0.2430, 0.2440, 0.2452, 0.3717, 0.3763, 0.4221, 0.4680, 0.5038, 0.5177, 0.6136, 0.6253, 0.6744, 0.8615, 1.3240.

Solving as above, we get $\tilde{\alpha}=3.5411221$, $\tilde{\beta}=3.9888550$, $\tilde{\lambda}=0.9282412$, $\tilde{\delta}=2.4304312$ and -2 ln L= -13.97776. Using this population as Y and above population as X, we get P = 0.6363636 and $\tilde{P}$ = 0.6181066.

For the case when α is unknown but β, λ and δ are known, we have conducted simulation experiments using bootstrap re-sampling technique for sample sizes n = 5, 10, 20 and 50. The samples are generated from (1.5), with α = 2.0, β = 0.5, λ = 1.5 and δ = 0.8. For different values of t, we have computed $\hat{R}(t), \tilde{R}(t)$, their corresponding bias, variance, 95% confidence length and corresponding coverage percentage. All the computations are based on 500 bootstrap replications and results are reported in Table 2.

In order to estimate 'P', for the case when α_1 and α_2 are unknown but other parameters are known, we have conducted simulation experiments using bootstrap re-sampling technique for sample sizes (n, m) = (5, 5), (10, 10), (20, 20), (25, 25), (50, 25) and (50, 50). The samples are generated from (1.5), with, $\alpha_1 = 1.0, \beta_1 = \beta_1 = 2.5, \lambda_1 = \lambda_2 = 1.5$ and $\alpha_2 = 1.6, 2.6, 4.6$ and 6.6. The computations are based on 500 bootstrap replications. We have computed $\hat{P}, \tilde{P}$, bias, variance, 95% confidence length and corresponding coverage percentage. The results are presented in Table 3.

5. Discussion

In the literature, the researchers have dealt with the estimation of R(t) and 'P', separately. If we look at the proofs of Theorems 2, 3, 4 and 5, we observe that the UMVUE and MLE of the sampled pdf is used to obtain the UMVUES and MLES of R(t) and 'P', respectively.

In Table 1, we compared UMVUE and MLE of α, keeping α, λ and δ to be constant for four-parameter exponentiated generalized

Lomax distribution. The table shows that UMVUE of α is more efficient than MLE of α. From table we observe that as we increase the sample size variance of estimators of α decrease (for both of estimators UMVUE as well as for MLE). Table 1 also shows that as we increase values of the parameter α, variance increases corresponding to both of the estimators.

With the help of Fig. 3, we justified the consistency property of the estimators. From Fig. 3, it also clear the UMVUES are more close to actual value than MLES.

Through Table 2, we compared the efficiency of $\hat{R}(t)$ and $\tilde{R}(t)$. Table 2 shows that UMVUE of R(t) is more efficient than MLE of R(t). It is also clear that as we increase sample size Biasness, MSE and Confidence Length decreases but on the other hand corresponding Coverage Percentage increases. These statements are also true for the estimators $\hat{P}$ and $\tilde{P}$.

6. Conclusion

We obtained UMVUES and MLES of parameter(s). UMVUES and MLES of R(t) and 'P' are derived. A comparative study of the two methods of estimation is done and we have established interrelationship between the two estimation problems. Moreover, in the present approach, one does not require the expressions of R(t) and 'P'.

It is observed that as the sample size increases, the average biases and mean squared errors decrease for all sets of parameters considered here [see Tables 2 and 3].

Table 2: Simulation results for R(t).

t	n	5		10		20		50	
	R(t)	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$
0.60	0.9351	0.9199 -0.0152 0.003740264 0.2085 89.3948	0.9061 -0.0290 0.003300959 0.1813 92.1337	0.9316 -0.0035 0.002305881 0.1838 90.1795	0.9252 -0.0099 0.002075589 0.1728 91.6010	0.9385 0.0034 0.001134942 0.1274 91.8912	0.9352 1e-04 0.001054313 0.1236 92.2744	0.9312 -0.0039 0.000592152 0.0994 93.7588	0.9325 -0.0026 0.0006014892 0.1009 93.6797
0.70	0.9196	0.8914 -0.0282 0.006681679 0.2867 90.3669	0.8827 -0.0369 0.005327485 0.2405 91.9609	0.9246 0.0050 0.002613866 0.1819 90.8083	0.9189 -7e-04 0.002241602 0.1727 92.1685	0.9191 -5e-04 0.001217573 0.1300 92.6667	0.9164 -0.0032 0.001129186 0.1248 92.8323	0.9223 0.0027 0.0007801377 0.1088 93.9131	0.9212 0.0016 0.000750414 0.1071 93.9680
0.80	0.9039	0.8316 -0.0723 0.02126833 0.4183 88.2364	0.8347 -0.0692 0.01591361 0.3585 89.5269	0.8968 -0.0072 0.004392456 0.2332 91.2278	0.8934 -0.0105 0.003767701 0.2177 92.1159	0.9007 -0.0032 0.002977223 0.1959 92.2869	0.8990 -0.0049 0.002752046 0.1889 92.5615	0.9060 0.0021 0.001049364 0.1225 93.6905	0.9052 0.0013 0.001009648 0.1204 93.7244
0.90	0.8883	0.8788 -0.0095 0.005605081 0.2729 91.8301	0.8721 -0.0162 0.003908982 0.2256 92.7269	0.8819 -0.0064 0.004433942 0.2369 91.9995	0.8796 -0.0087 0.003722797 0.2180 92.4491	0.8867 -0.0016 0.002731544 0.2010 93.8816	0.8855 -0.0028 0.002498929 0.1925 94.0065	0.8902 0.0019 0.0007905776 0.1067 93.9069	0.8897 0.0014 0.0007590906 0.1047 93.9190
1.00	0.8729	0.8614 -0.0115 0.01406574 0.3400 83.4812	0.8639 -0.0090 0.006595676 0.2847 91.0036	0.8691 -0.0038 0.004710623 0.2527 93.1672	0.8680 -0.0049 0.003921974 0.2321 93.4761	0.8755 0.0026 0.001893337 0.1655 93.7172	0.8748 0.0019 0.001712915 0.1577 93.7658	0.8742 0.0013 0.001170059 0.1332 94.1217	0.8745 0.0016 0.00121732 0.1358 94.1142
1.10	0.8578	0.8120 -0.0458 0.009230732 0.3060 92.7486	0.8178 -0.0399 0.00622321 0.2488 92.9063	0.8664 0.0087 0.006022213 0.2899 93.4050	0.8657 0.0080 0.005033709 0.2695 94.0357	0.8621 0.0043 0.00259252 0.1928 94.1945	0.8620 0.0043 0.002354974 0.1839 94.2313	0.8597 0.0020 0.00105026 0.1278 95.1719	0.8598 0.0020 0.001009774 0.1253 95.1754
1.20	0.8429	0.8345 -0.0084 0.02373229 0.4143 80.5189	0.8489 0.0060 0.009829836 0.3463 90.0176	0.8407 -0.0022 0.008419463 0.3362 92.9148	0.8424 -6e-04 0.007046862 0.3118 93.3779	0.8445 0.0015 0.003354482 0.2188 94.0666	0.8452 0.0023 0.003049492 0.2088 94.0984	0.8444 0.0014 0.00138267 0.1408 94.1825	0.8447 0.0018 0.001330868 0.1381 94.1854

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

Table 3: Simulation results for 'P'.

(α_1, α_2)	**(1, 1.6)**		**(1, 2.6)**		**(1, 4.6)**		**(1, 6.6)**	
P	0.3846154		0.2777778		0.1785714		0.1315789	
(n, m)	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$
(5, 5)	0.3786 -0.0060 0.01179814 0.3983 91.3746	0.3912 0.0066 0.009654342 0.3605 91.7915	0.2756 -0.0021 0.00564374 0.2905 93.3992	0.2961 0.0183 0.005363467 0.2749 93.9089	0.177 -0.0016 0.006724384 0.3054 89.9674	0.1974 0.0188 0.007080985 0.3065 91.0873	0.1332 0.0017 0.002085573 0.1707 90.2432	0.1532 0.0216 0.002741058 0.180 91.1396
(10, 10)	0.3803 -0.0043 0.006770484 0.3101 93.5394	0.3861 0.0015 0.006166892 0.2968 93.6570	0.2667 -0.0111 0.004350609 0.2493 93.7909	0.2762 -0.0016 0.004049913 0.2443 93.9623	0.1688 -0.0098 0.004009771 0.2660 93.7235	0.1783 -3e-04 0.003959255 0.2678 94.1629	0.1324 8e-04 0.001335336 0.1442 94.3738	0.1414 0.0099 0.001518139 0.1486 94.4979
(20,20)	0.3848 2e-04 0.003831629 0.2327 93.9806	0.3875 0.0029 0.003681852 0.2279 93.9843	0.2771 -6e-04 0.0027125 0.1960 93.9572	0.2816 0.0039 0.002672573 0.1940 93.9734	0.1747 -0.0039 0.001455662 0.1525 94.0237	0.1795 9e-04 0.001454373 0.1532 94.1187	0.1318 2e-04 0.0007120453 0.1063 94.3780	0.1361 0.0045 0.0007538388 0.1079 94.4327
(25,25)	0.3849 2e-04 0.002434507 0.1923 94.8644	0.3871 0.0025 0.002358028 0.1891 94.8740	0.2771 -6e-04 0.001743998 0.1624 94.7293	0.2808 0.0030 0.001722742 0.1611 94.7503	0.1775 -0.0010 0.0009275121 0.1182 94.5227	0.1814 0.0028 0.0009425864 0.1187 94.5524	0.1380 0.0065 0.0006626313 0.0985 94.7358	0.1416 0.0100 0.0007342276 0.0996 94.7668
(50,25)	0.3858 0.0012 0.002148201 0.1818 95.0124	0.3851 5e-04 0.002083882 0.1792 95.0139	0.2709 -0.0069 0.001668848 0.1571 94.8713	0.2716 -0.0062 0.001628462 0.1555 94.8678	0.1752 -0.0034 0.0008704789 0.1152 94.9678	0.1766 -0.0020 0.0008588856 0.1149 94.9756	0.1315 -1e-04 0.0004882573 0.0869 94.9388	0.1329 0.0013 0.0004921131 0.0871 94.9468
(50, 50)	0.3822 -0.0024 0.001999618 0.1769 95.1724	0.3834 -0.0013 0.001962234 0.1754 95.1780	0.2765 -0.0012 0.001498307 0.1533 95.2211	0.2783 6e-04 0.00148525 0.1527 95.2254	0.1776 -9e-04 0.0007962493 0.1117 95.1364	0.1795 9e-04 0.0008000788 0.1120 95.1439	0.1308 -8e-04 0.0004820625 0.0869 95.0744	0.1325 9e-04 0.0004883238 0.0875 95.0824

Here, the first row indicates the estimate, the second row indicates the bias, the third row indicates variance, the fourth row indicates 95% bootstrap confidence length and the fifth row indicates the coverage percentage.

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